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TOPICAL ISSUE

On the Boundary Value of the Time-Delay and the Asymptotic Behavior of a Continuous First-Order Consensus Protocol

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Abstract—Coordination in multiagent systems with information influences is studied. In particular, a model of multiagent system in which information is transmitted with a constant delay for all agents is studied. Using the Nyquist criterion applied by Tsypkin for systems with delayed feedback, a formula is obtained for the boundary value of time-delay which is included as a parameter in the system of differential equations with an asymmetric constant Laplacian matrix. The condition of independence of stability from time-delay is founded. The results obtained generalize some previously results and can be applied in coordination analysis in a multiagent systems with complex protocol.

Keywords: multiagent system, consensus, eigenprojection, Laplacian matrix, communication digraph, stability analysis with delay, Tsypkin's test

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1. INTRODUCTION

Since the 40s of the last century, there have been many publications and classical results on the stability of quasi-polynomials, functional-differential equations, differential equation with a deviating argument, etc. (see [1–6]). Time-delayed systems are also well-studied part of control theory (see [7–13]).

The problem of time-delay in multiagent systems with information influences has been studied by a number of authors in papers dedicated to consensus (see [14–20]). The second part of the book [14] provides an overview of some papers dedicated to delay in discrete-time systems, synchronization in networks with delayed connections, approximate consensus in networks with measurement delays, etc.

In this paper coordination protocol in multiagent systems (MAS) is studied. It is assumed that agents receive data on the states of their neighbors with delay. Also agents average their own data without delay using the same topology to coordinate characteristics. Problem of dependence of boundary value of time-delay on the spectrum of Laplacian matrix is solved for such protocol. Such task is actual for multiagent system models as well as systems with an arbitrary square matrix. Solution of this problem is reduced to solving scalar equation with real coefficient for symmetric matrices, which is well-studied. However, similar scalar equation with complex one for an arbitrary square matrices is not studied. The study is also of theoretical interest. Obtained results may be used for analysis of systems with multiple inputs and multiple outputs (MIMO-systems).[1]

[1] Some above mentioned results are briefly presented at the RusAutoCon-2023 conference without evidence.

MAS with information influences occupy a central place among all MAS encountered in various subject areas (see [21–26]). The peculiarity of such systems is that the system with information influences, as usual, is represented as a directed graph, the vertices of which correspond to the agents, and the arcs correspond to the influences of the agents on each other. When applying the continuous protocol, the Laplacian matrix is constructed and the system is described by a system of differential equations with the constructed Laplacian matrix.

The paper has the following structure. Section 2 presents the basic concepts and definitions used in the work, presents some well-known results on boundary values of the time-delay, and provides auxiliary and previously proven propositions. Section 3 presents the main results of this paper and their corollaries.

2. NECESSARY TERMS AND AUXILIARY RESULTS

Consider a multiagent system with a set of agents $V = \{1, \dots, n\}$. Such system can be presented as a communication digraph $\Gamma = (V, E)$, where $E \subset V \times V$ is the set of arcs. The set of agents is the set of vertices of Γ in this notation. It is assumed that if agent j influences agent i with weight a_{ij}, then there is an arc from vertex j to vertex i with weight a_{ij}. The matrix $A = (a_{ij})$ will be called a communication matrix (or matrix of influences).

Definition 1. L is the Laplacian matrix, corresponding to Γ, defined as follows: if $i \neq j$, then $l_{ij} = -a_{ij}$ and $l_{ii} = \sum_{k \neq i} a_{ik}$ otherwise.

By definition of the Laplacian matrix $L\mathbf{1}_n = \mathbf{0}_n$, where $\mathbf{1}_n$ and $\mathbf{0}_n$ are vectors of dimension n, consisting of ones and zeros respectively. It means, that 0 is an eigenvalue of L. It is easy to establish (for example by Gershgorin's theorem), that every nonzero eigenvalue of L have a positive real part.

Definition 2. The index ν of a square matrix A is the order of greatest Jordan block with zero diagonal in the Jordan form of A or the minimal number ν satisfied the equality $\operatorname{rank}(A^{\nu}) = \operatorname{rank}(A^{\nu+1})$.

Definition 3. The eigenprojection (see, for example, [27]) of a square matrix A is a stochastic idempotent matrix $A^{\vdash}$, such that $\mathcal{R}(A^{\vdash}) = \mathcal{N}(A^{\nu})$ and $\mathcal{N}(A^{\vdash}) = \mathcal{R}(A^{\nu})$, where ν is the index of A, $\mathcal{R}(X) = \{y : y = Xx\}$ is range of X, $\mathcal{N}(X) = \{x : Xx = 0\}$ is the kernel of X.

Definition 4. We will say that a MAS with time-delay and a given protocol is stable if there exists a finite limit of the vector of agent's characteristics.

Definition 5. We will say that a MAS with time-delay and a given protocol converges if consensus is reached for any vector of initial states, or if the limit of vector of agent's characteristics may be presented as $c\mathbf{1}_n$, where $c \in R^1$ is consensus value.

In this paper we will consider the consensus protocol in multiagent systems in the form

$$\dot{x}(t) = -aLx(t) - bLx(t - \tau), \tag{1}$$

where $a \geqslant 0$, $b > 0$, $a \neq b$.

The protocol (1) is a special case of a time-delay system:

$$\dot{x}(t) = A_1 x(t) + A_2 x(t - \tau), \tag{2}$$

for which two problems are often studied: 1) under what condition is the stability of the system independent of delay; 2) if the stability depends on the delay, then find the critical value of the time-delay.

Let I be a identity matrix. The characteristic function of the system (2) is defined as follows:

$$f(s) = \det(sI - A_1 - A_2 e^{-s\tau}). \tag{3}$$

It is known that the function $f(s)$ is entire (analytic in the entire complex plane), and all its zeros on the complex plane are located as follows: there exists a real number η such that there are no zeros s with $\mathrm{Re}(s) > \eta$ and only a finite number of zeros with $0 \leqslant \mathrm{Re}(s) \leqslant \eta$. The system (2) has been studied by many authors (see, e.g. [1, 5, 6, 9–12] and the bibliography in [9]). A sufficient condition for the stability of the system (2) depending on the critical value τ is given in [9] using the linear matrix inequality and the Lyapunov method. The dependence of the stability from τ through the generalized eigenvalues A_1 and A_2 is studied in [12]. A lot of papers are devoted to the independence of the stability of the system from the delay value. For example, in [12] it is proved that if the matrices A_1 and $A_1 + A_2$ in system (2) are stable, and the condition

$$\rho((i\omega I - A_1)^{-1}A_2) < 1, \ \forall \omega > 0 \tag{4}$$

is satisfied, then the system (2) is stable.

In [10] it was investigated the effects induced by the delay parameter on the stability of linear dynamical systems with time-delay. For this purpose authors proposed a frequency-sweeping framework.

In [11] the properties of the spectrum of a quasi-polynomial (2) are given and the relationship between the maximum permissible multiplicity of eigenvalues and the spectral abscissa $-$ of the largest real part of the spectrum of the matrix of a dynamical system is studied.

The dependence of the critical value of delay on the spectrum of Laplacian matrix is of great interest and is a non-trivial problem. However, relatively few works are devoted to consensus protocol with time-delay. Nevertheless, as noted above, several sections of book [14] are devoted to certain aspects of network control with delay.

In [23] basic time-delay protocol for symmetrical case was first considered:

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(x_i(t-\tau) - x_j(t-\tau)), \tag{5}$$

where $\mathcal{N}_i$ is set of agents j, that influence agent i with weight a_{ij}.

The consensus condition for such a protocol (with a symmetric graph) is reduced to the scalar case and has been repeatedly deduced by different authors. For example, from [1] this result can be obtained as a special case. The stability condition (5) is also found for an arbitrary stable matrix in [28].

Another, no less realistic protocol was discussed in [29]:

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(x_i(t) - x_j(t-\tau)), \tag{6}$$

or in matrix form

$$\dot{x}(t) = -\mu I x(t) + Ax(t-\tau), \tag{7}$$

where μ is the sum of the row elements of A, which is considered the same for all rows of A.

It is known that the reachability of consensus in the protocol (5) depends on L, i.e. system (5) is stable, if τ is not greater than some boundary value depending on the spectrum of L. In addition, if consensus is reached in the system (5), then its value does not depend on τ. However, the system (6) is stable for all τ. But if consensus is reached too, its value can be depended on τ (see Theorem 2 in [29]).

Next, consider protocol (1). Note that this system is being investigated using the scalar equation

$$\dot{y}(t) = -\alpha y(t) - \beta y(t-\tau), \tag{8}$$

where $\alpha = a\lambda$, $\beta = b\lambda$, λ is eigenvalue of L. The equation (8) for the real numbers α and β has been studied by many authors. For example, back in 1950, [30] studied the roots of the transcendental function, which is the characteristic function of a scalar difference-differential equation. (Also see [1] and Example 2.4 in [12]).

More general case

$$\dot{y}(t) - c\dot{y}(t-\tau) = -ay(t) - by(t-\tau) \tag{9}$$

was studied in [9], where the problem of stability depending on the real numbers a, b and c was considered.

We study the system (1) using a scalar equation (8), where α and β are complex numbers.

Note that in the case of $a = 0$, the protocol (1) becomes the basic consensus protocol with time-delay (matrix form (5)):

$$\dot{x}(t) = -Lx(t-\tau). \tag{10}$$

To solve the problems under study, we will need the following well-known results, which are closely related to the theory of multiagent systems.

Proposition 1 [31]. *For any Laplacian matrix L its index* (*see the definition above*) *is equal to 1, i.e.* $\operatorname{ind} L = 1$, *and*

$$LL^{\vdash} = L^{\vdash}L = 0_{n\times n}. \tag{11}$$

The following theorem has been proved in [31, Theorem 5].

Theorem 1. $L^{\vdash} = \lim_{t\to\infty}(I + tL)^{-1}$.

Proposition 2 [32]. *If $x(t)$ is solution of system* (12)

$$\dot{x}(t) = -Lx(t), \tag{12}$$

then

$$\lim_{t\to\infty} x(t) = L^{\vdash}x(0). \tag{13}$$

Proposition 3. *Let the system* (10) *be stable, and let $x(t)$ be a solution of* (10). *Then* (13) also holds for solution of (10). *Moreover, if* 0 *is a simple eigenvalue of L, then the protocol* (10) *converges.*

Proposition 3 is easily proven using the final value theorem, Proposition 2 and Theorem 1.

3. MAIN RESULTS

Consider the following problems for the system (1): 1) at what values a and b is the system (1) stable regardless of τ; 2) if the stability of the system depends on τ, then find boundary value of τ; 3) if consensus is reached for any vector of initial states, then what is the consensus value?

The characteristic function of the system (1) is a transcendental function and has the form

$$f(s) = \det(sI + aL + be^{-\tau s}L). \tag{14}$$

If $\lambda_j \in \sigma(L)$ is an eigenvalue of L, then $f(s)$ can be represented as

$$f(s) = \prod_{j=1}^{n} f_{\lambda_j}(s) = \prod_{j=1}^{n}(s + a\lambda_j + b\lambda_j e^{-\tau s}).$$

Let $f_{\lambda_j}(s)$ be the characteristic function of the system for the scalar case:

$$f_{\lambda_j} = Q(s) + P(s)e^{-\tau s} = s + a\lambda_j + b\lambda_j e^{\tau s} = 0, \quad j = 1, \ldots, n, \tag{15}$$

where $Q(s) = s + a\lambda_j, P(s) = b\lambda_j$.

The stability of quasi-polynomial is studied by Tsypkin's test [7]), based on Nyquist stability criterion for delayed case.[2] For this purpose a quasi-polynomial $Q(s) + P(s)e^{-\tau s}s$ is taken as characteristic function of a system with delayed feedback. The stability of one is estimated by the transfer function of corresponding open-loop system with delay.

Let us represent the open-loop transfer function as (for more details, see [7])

$$W_\tau(i\omega) = -\frac{P(i\omega)}{Q(i\omega\tau)}e^{-i\tau\omega} = -\frac{b\lambda_j}{i\omega + a\lambda_j}e^{-i\tau\omega} = -\frac{\beta}{i\omega + \alpha}e^{-i\tau\omega}, \tag{16}$$

where $\alpha = \alpha_1 + i\alpha_2 = a\lambda_j, \beta = \beta_1 + i\beta_2 = b\lambda_j$. For simplicity we omit the subscript for α and β which indicated the jth eigenvalue of matrix L.

Further the argument of λ_j is denoted by ϕ_j, i.e. assume $\phi_j = \arg(\lambda_j)$. Let

$$\begin{aligned} W(i\omega) &= -\frac{P(i\omega)}{Q(i\omega\tau)} = -\frac{\beta_1 + i\beta_2}{\alpha_1 + i(\omega + \alpha_2)} = -\frac{(\beta_1 + i\beta_2)(\alpha_1 - i(\omega + \alpha_2))}{\alpha_1^2 + (\omega + \alpha_2)^2} \\ &= -\frac{\beta_1\alpha_1 + \beta_2(\omega + \alpha_2)}{\alpha_1^2 + (\omega + \alpha_2)^2} - i\frac{\alpha_1\beta_2 - \beta_1(\omega + \alpha_2)}{\alpha_1^2 + (\omega + \alpha_2)^2} = x(\omega) + iy(\omega) = W^0(\omega)e^{i\theta(\omega)}, \end{aligned} \tag{17}$$

and

$$W_\tau(i\omega) = W(i\omega)e^{-i\tau\omega} = W^0(\omega)e^{i\theta(\omega)}e^{-i\tau\omega}. \tag{18}$$

If $W(i\omega) = x(\omega) + iy(\omega)$, then after some transformations it can be shown that

$$\left(x(\omega) + \frac{b}{2a}\right)^2 + \left(y(\omega) + \frac{b\,\mathrm{Im}\,(\lambda_j)}{2a\mathrm{Re}(\lambda_j)}\right)^2 = R^2, \tag{19}$$

where $R = \frac{b|\lambda_j|}{2a\mathrm{Re}(\lambda_j)}$, i.e. function $W(s)$ conformal maps an imaginary axis into the circle.

The next proposition follows from (17) and the identity (19).

Proposition 4. *The transfer function $W(i\omega)$ is a circle on the complex plane centred at the point $\left(-\frac{b}{2a}; -\frac{b\,\mathrm{Im}\,(\lambda_j)}{2a\mathrm{Re}(\lambda_j)}\right)$ with radius $R = \frac{b|\lambda_j|}{2a\mathrm{Re}(\lambda_j)}$.*

Note that

$$y = \frac{\beta_1\omega}{\alpha_1^2 + (\omega + \alpha_2)^2}. \tag{20}$$

According to [7] the expression (17) will be called the transfer function of open-loop *equivalent* system without delay. The stability of the open-loop equivalent system follows from $Q = s + a\lambda$, where $a > 0$, and the real parts of λ are positive. Let us represent the transfer function of closed-loop system with delay as

$$\frac{W_\tau(i\omega)}{1 - W_\tau(i\omega)}.$$

By virtue of the Nyquist's criterion, an equivalent system is stable if the point $(1, i0)$ lies outside the Nyquist plot, and unstable otherwise.

The next proposition provides a sufficient condition for the stability of the system (1).

[2] This method is closely related with argument principle.

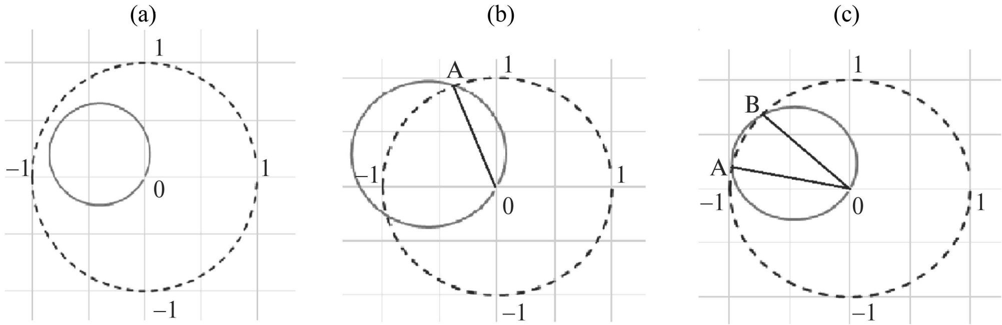

Fig. 1.

Proposition 5. *System* (1) *is stable for all* $\tau \geqslant 0$, *if*

$$\frac{a}{b} > \max_{\lambda_j \in \sigma(L) \setminus \{0\}} \frac{1}{\cos \phi_j}. \tag{21}$$

Remark 1. The proof of Proposition 5 can also be obtained from famous Polyak's method of stability and robust stability of uniform system consisting of identical elements and amplifiers, published in [33]. According to [33], if $W(s) = \frac{A(s)}{B(s)}$ is proper function (degree of $A(s)$ no more than degree of $B(s)$) and has no right poles, then for stability of $D(W(s))$ all roots of $D(p)$ must lie outside the circle of radius $\|W(s)\|_\infty$:

$$\|p_i\| > \|W(s)\|_\infty. \tag{22}$$

In this case $W(s) = -\frac{P(s)}{Q(s)}$ is proper function and has no right poles, and $D(p) = 1 - p$. According to the Proposition 5 $\|W(s)\| < 1$ for all s, i.e. condition (22) is fulfilled.

Remark 2. Another proof of the Proposition 5 can be obtained as corollary from Theorem 2.1 in [12] (see, for Example 2.6 ibidem) by applying it to the system (1).

In [34] the generalized frequency method, similar to the frequency analysis method [33] is used for the analysis of a multiagent system with interacting agents. Also in [34] in Example 2 inertial element with delay was considered. However, for its stability generalized frequency method was applied for fractional rational function that is not transcendental.

Note that when the inequality (21) is satisfied, the open-loop transfer function strictly belongs to the unit circle centered at $(0, 0)$ (Fig. 1a).

Theorem 2. *Let condition* (21) *is not fulfilled and*

$$\lambda_j \in \Lambda = \left\{ \lambda \in \sigma(L) \setminus \{0\} \mid a \leqslant \frac{1}{\cos \phi_j} b \right\}.$$

Then real parts of the roots of the quasi-polynomial (15) *is negative if*

$$\tau < \tau_0^j = \frac{\arccos\left(-\frac{\beta_1 \alpha_1 + \beta_2 \sqrt{|\beta|^2 - \alpha_1^2}}{|\beta|^2}\right)}{\sqrt{|\beta|^2 - \alpha_1^2} - \alpha_2}, \tag{23}$$

and system (1) *is stable for all* $\tau < \tau_0$, *where* $\tau_0 = \min_{\lambda_j \in \Lambda} \tau_0^j$.

According to [7], if $W(i0) < -1$ (Fig. 1b), there is one critical frequency. In this case $\frac{a}{b} < 1$, the critical value of time-delay is determined in a unique way and it is boundary one. If $\tau < \tau_0^j$, then quasi-polynomial (15) is stable, and if $\tau > \tau_0^j$, then one is unstable.

If conditions of Theorem 2 are fulfilled and $|W(i0)| < 1$, then transfer function intersects the unit circle twice, and there are two critical frequency ω_1 and ω_2, where $\omega_1 < \omega_2$, corresponding to points A and B from Fig. 1c. Two critical values of time-delay τ_1 and τ_2 are determined by two critical frequency respectively. In this case the system is stable if $\tau < \tau_2$, and τ_2 is boundary value of system stability. If $\tau_1 > \tau > \tau_2$, then the system is unstable. If $\tau > \tau_2$, then a further increase of τ leads to an alternating instability and stability of the system. If points A and B are sufficiently close, i.e. difference between τ_1 and τ_2 are sufficiently small, then an arbitrary number of stability and instability intervals are possible.

Corollary 1. *If L is symmetric Laplacian matrix, $b = 1$ and $a = 0$, then from (23) for each eigenvalue we obtain:*

$$\tau_0^j = \frac{\arccos 0}{\lambda_j} = \frac{\pi}{2\lambda_j}.$$

Note that $\tau_0 = \frac{\pi}{2\lambda_{\max}}$. This result is the same as the result given in [23]. However, for the scalar case, this estimate was given in [7] as an example and was repeatedly obtained in various literature.

Let in (23) $b = 1$ and $a = 0$. Then for the eigenvalue λ_j due to negativeness $\beta_2 = b\,\mathrm{Im}\,(\lambda_j) = \mathrm{Im}\,(\lambda_j)$ there executes

$$\tau_0^j = \frac{\arccos\left(-\frac{\beta_2}{|\beta|}\right)}{|\lambda_j|} = \frac{\arccos(\sin|\phi_j|)}{|\lambda_j|} = \frac{\pi/2 - |\phi_j|}{|\lambda_j|}.$$

Corollary 2. *If L is an arbitrary Laplacian matrix, $b = 1$ and $a = 0$, then*

$$\tau_0 = \min_{\lambda_j \neq 0} \frac{1}{|\lambda_j|}\left(\frac{\pi}{2} - |\phi_j|\right).$$

This expression coincides with the result on the stability of an arbitrary matrix obtained in [28].

Corollary 3. *If $L = 1$, i.e. the scalar case of the equation (1) is considered, then from (23) we obtain:*

$$\tau_0 = \frac{\arccos(-\frac{ab}{b^2})}{\sqrt{b^2 - a^2}} = \frac{\arccos(-\frac{a}{b})}{\sqrt{b^2 - a^2}}.$$

The resulting expression coincides with the boundary value τ for the scalar equation (2.22) from [1] (see Example 2.4 from [1]).

Theorem 3. *Let system (1) is stable. Then if $x(t)$ is solution of following system*

$$\begin{cases} \dot{x}(t) = -aLx(t) - bLx(t-\tau), & t \geqslant 0; \\ x(t) = 0, & t \in [-\tau, 0), \end{cases}$$

then

$$\lim_{t\to\infty} x(t) = L^{\vdash} x(0),$$

i.e. the vector to which protocol (1) converges, does not depend on the coefficients a and b, as well as on τ.

Corollary 4. *If the condition of the Theorem 3 is fulfilled and 0 is a simple eigenvalue of L, then the protocol (1) converges to a consensus whose value is equal to the value of the base consensus protocol $\dot{x}(t) = -Lx(t)$.*

4. CONCLUSION

In this paper consensus protocol with time-delay has been studied. A condition under which convergence of consensus protocol does not depend on time-delay has been obtained. In case of violation of this condition formula for boundary value of time-delay has been founded. The asymptotic behavior of this protocol has also been studied. It is proved that if consensus is reached in a multiagent system with time-delay for any vector of initial states, then it is determined by product of eigenprojection and the vector of initial states. The expression for the boundary value τ obtained in this paper generalizes some previously obtained formulae. A further object of study by the authors is a consensus model in the form (2), according to which agents first receive data from their immediate neighbors without delay, and then with some delay τ from other agents.

APPENDIX

A.1. PROOF OF PROPOSITION 5

If the condition (21) is satisfied, then for any $\lambda_j \in \sigma(L) \setminus \{0\}$ it is true

$$\frac{a}{b} > \frac{1}{\cos\phi_j}.$$

From the last inequality it follows:

$$\frac{a^2}{b^2} > \frac{\operatorname{Re}^2(\lambda_j) + \operatorname{Im}^2(\lambda_j)}{\operatorname{Re}^2(\lambda_j)} \Longrightarrow \alpha_1^2 > |\beta|^2.$$

Then

$$|W(i\omega)| = \sqrt{\frac{\beta_1^2 + \beta_2^2}{\alpha_1^2 + (\omega + \alpha_2)^2}} < \sqrt{\frac{|\beta|^2}{|\beta|^2 + (\omega + \alpha_2)^2}} \leqslant 1.$$

Therefore if $\frac{a}{b} > \max_{\lambda_j \in \sigma(L)\setminus\{0\}} \frac{1}{\cos\phi_j}$, then $|W_0(i\omega)| < 1$, and system (1) is stable for all τ.

A.2. PROOF OF THEOREM 2

The stability problem of the system with protocol (1) will be studied by a quasi-polynomial (15) with λ_j or $\bar{\lambda}_j$. The system (1) is stable, if for each $\lambda_j \in \Lambda$ the corresponding quasi-polynomial (15) is stable.

So, for a fixed λ_j we study the stability of the quasi-polynomial (15).

An increase in the delay τ can lead to the fact that the point $(1, 0)$ will lie inside Nyquist plot $W_\tau(i\omega)$. The value τ at which the zeros of the function (15) cross the imaginary axis or at which the point $(1, 0)$ belongs to $W_\tau(i\omega)$ is called critical. Such times and frequencies are determined by the condition

$$W_\tau(s) = 1, \tag{A.1}$$

or

$$W^0(\omega) e^{i\theta(\omega)} e^{-i\tau\omega} = 1. \tag{A.2}$$

To fulfillment of (A.2) there must be:

$$1) \quad W^0(\omega) = \frac{|\beta|}{\sqrt{\alpha_1^2 + (\omega + \alpha_2)^2}} = 1; \quad 2) \quad \theta(\omega) - \tau\omega = -2\pi n,$$

where n is a positive integer number.

Condition $|W^0(\omega)| = 1$ is fulfilled when $(\omega + \alpha_2)^2 = |\beta|^2 - \alpha_1^2$, i.e.

$$\omega = \pm\sqrt{|\beta|^2 - \alpha_1^2} - \alpha_2. \tag{A.3}$$

Then

$$x = -\frac{\beta_1\alpha_1 + \beta_2(\omega + \alpha_2)}{\alpha_1^2 + (\omega + \alpha_2)^2} = -\frac{\beta_1\alpha_1 \pm \beta_2\sqrt{|\beta|^2 - \alpha_1^2}}{|\beta|^2}; \tag{A.4}$$
$$y = -\frac{\alpha_1\beta_2 - \beta_1(\omega + \alpha_2)}{\alpha_1^2 + (\omega + \alpha_2)^2} = -\frac{\alpha_1\beta_2 \pm \beta_1\sqrt{|\beta|^2 - \alpha_1^2}}{|\beta|^2}.$$

From second condition, i.e. $\theta(\omega) - \tau\omega = -2\pi n$, follows:

$$\cos\theta(\omega) = \cos(\tau\omega - 2\pi n);$$
$$\cos\theta(\omega) = \cos\tau\omega. \tag{A.5}$$

Let us consider in more detail $\theta(\omega)$. In virtue of $x^2 + y^2 = 1$ (by virtue of the fact that (x, y) is the point of intersection with unit circle) $\cos\theta(\omega) = x$, and from (A.3), (A.4) and (A.5) we obtain:

$$\tau_0^j = \frac{\arccos x}{\omega} = \frac{\arccos\left(-\frac{\beta_1\alpha_1 \pm \beta_2\sqrt{|\beta|^2 - \alpha_1^2}}{|\beta|^2}\right)}{\pm\sqrt{|\beta|^2 - \alpha_1^2} - \alpha_2}. \tag{A.6}$$

If λ_j is a complex eigenvalue of the Laplacian matrix L, then stability (15) is studied for λ_j and $\bar{\lambda}_j$. The denominator of the right-hand side (A.6) reaches its maximum value when the imaginary part of the pair of eigenvalues under consideration is negative and the sign in front of the root is positive, i.e. for $\omega = \sqrt{|\beta|^2 - \alpha_1^2} + |\alpha_2|$. Similarly, the numerator is a decreasing non-negative function: the larger the value $\left(-\frac{\beta_1\alpha_1 \pm \beta_2\sqrt{|\beta|^2 - \alpha_1^2}}{|\beta|^2}\right)$, the smaller the value of the function $\arccos(x)$. Therefore, the minimum value of τ_0^j is achieved at the eigenvalue with a negative imaginary part, and the boundary value is given by (23). Then the delay boundary value for (1) is defined as $\tau_0 = \min_{\lambda_j \in \Lambda} \tau_0^j$.

A.3. PROOF OF THEOREM 3

Let the condition for the boundary value τ be satisfied. Then $x(t)$ will have a limit, and by the final value theorem we get:

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s(sI + aL + be^{-s\tau}L)^{-1}x(0)$$
$$= \lim_{s\to 0}\left(I + \frac{1}{s}(a + e^{-s\tau})L\right)^{-1} x(0) = \lim_{t\to\infty}\left(I + t(a + be^{-\tau/t})L\right)^{-1} x(0). \tag{A.7}$$

According to theorem 1 we have

$$\lim_{t\to\infty}\left(I + t(a + be^{-\tau/t})L\right)^{-1} = \lim_{t\to\infty}(I + tL)^{-1} = L^{\vdash}.$$

Then from (A.7) we get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(I + t(a + be^{-\tau/t})L\right)^{-1} x(0) = L^{\vdash}x(0).$$

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